where they are taught to count and calculate based on the decimal system without being proffered any explanation of the principles on which that system is founded.

Decimal System Demonstration Material

The material that we provide to children, to understand the decimal system, is threefold and composed of objects, numerical symbols and words.

The objects are coloured beads.[15]

For now we will only consider what serves to demonstrate in practical terms the structure of the decimal system.

The objects consist of individual beads and small bars, consisting of a metal wire on which ten beads are threaded and fixed.

Next there are squares of beads, made up of ten of the bead bars described, attached to each other in such a way as to form a single object, a "square of ten", in other words a hundred beads. And finally, a cube is constructed by placing ten of the previous squares one over another, linked together and constituting a single object (Figure 8). This single cube of beads is a point of arrival and a limit to the part of the system introduced here (Figure 9).

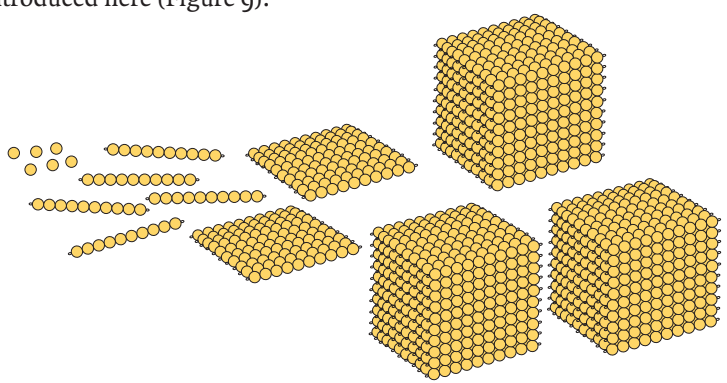

Figure 8
Material of the decimal system (elements)

15 Beads are golden, all the same colour.

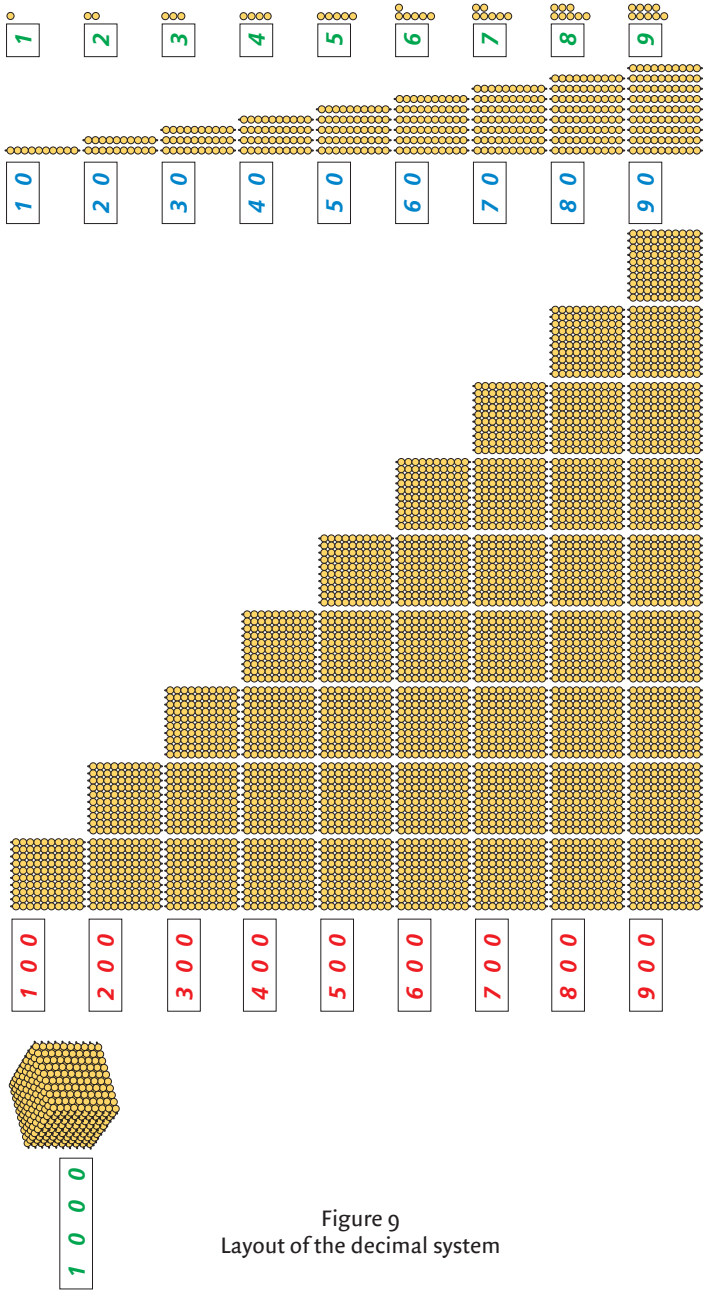

Figure 9
Layout of the decimal system

Other parallel exercises on the decimal system

Snake game

An exercise that can be undertaken parallel to the previous exercise, and whose purpose is to help children carry out small additions of units almost mechanically, preparing them for mental arithmetic, makes use of the bead material representing numerical groups of less than ten: 1, 2, 3, 4, 5, 6, 7, 8 and 9. This consists of beads threaded onto a metal wire, each group in a different colour (Figure 17).

A large quantity of bead bars is necessary for this exercise. The number represented by each bar is found by counting the beads on the bar. Little by little, however, the colour will help with recognition of the quantity, doing away with the task of having to count the beads one by one. In this way, the number that each bar represents is visible at first sight, due to its characteristic colour. This large amount of bead bars is a mixture of only nine digits.

The exercise begins by lining up bead bars, randomly chosen. The bars are aligned carefully, either on a long table or on the floor and, in order to avoid using too much space, the line is sinuous, reminiscent of the movement of a snake (Figure 25).

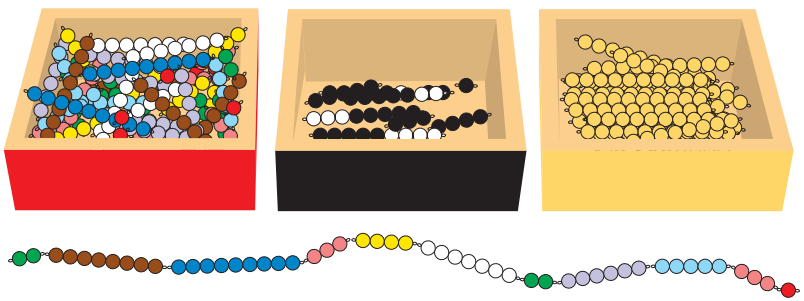

Figure 25
The snake and the search for 10

Counting then starts and, as soon as the number ten has been reached, the bars added together are set aside and are replaced with a bar of ten beads that, as we know, is golden. From there count another ten, and behold, another ten bar replaces the bars added and so on until the end. We observe the following transformation: little by little the colour gold eats up the multi-coloured line (Figure 25), and bars of equal length gradually take the place of those that were previously all of differing lengths (Figure 26).

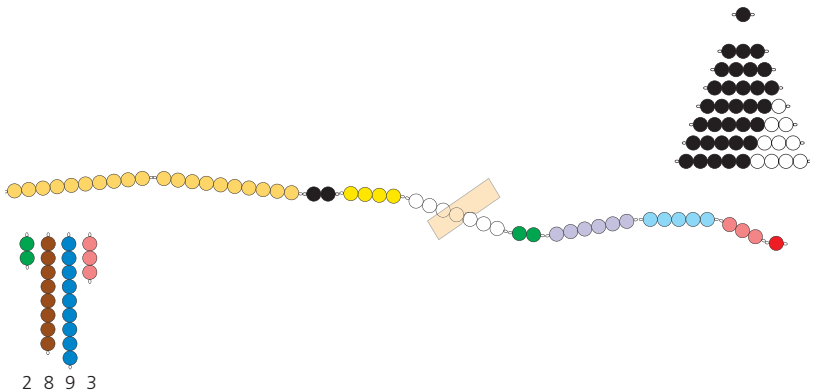

Figure 26
The snake, some 10s counted

In this case, the count is made to transform smaller amounts into tens, the base of the decimal system. The exercise therefore is to make tens with the smaller amounts. At ten, the count begins again and the already counted tens accumulate along the way. It is an unvarying work that is repeated and concludes by making it easy, fast and mechanical to compute sums of digits less than ten. Memorization, however, is achieved through a long process of counting units to ten, forcing one to reflect and carry out a number of small operations; that is, subtractions performed at the same time, in order to work out the quantity in excess of ten once that ten has been formed. Based on this special feature, the exercise is developed in all its varieties arising from the possible combinations of different bead bars in the formation of the snake.

We shall now describe in detail the manner in which the exercise is performed.

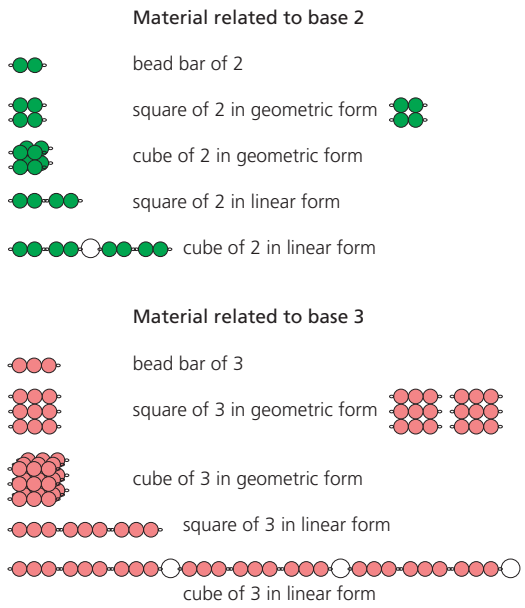

Figure 155
Bar, squares, cube, square and cube chain, for numbers 2 and 3

Having enough space to display all the chains in parallel and fully extended, a realistic knowledge of the sequence of the squares of the numbers can be achieved.

The rigid shape of the squares, where beads are linked, allows their superimposition, which highlights the relationship between their numerical size.

1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2
1	4	9	16	25	36	49	64	81	100

Comparing in the same way the rigid form of the cubes and the cube chains, we observe the great difference between the square and cube of a number.

1^3	2^3	3^3	4^3	5^3	6^3	7^3	8^3	9^3	10^3
1	8	27	64	125	216	343	512	729	1000

This gives a sensorial view of the relative lengths of the square and the cube of the numbers and their sequence.
Comparing the rigid squares and the cubes, it will be seen how the third dimension develops, starting from each square. Each cube can be superimposed on its square and vice versa, but the difference between the square and the cube lies in the third dimension. The fact that for each number there is a sufficient quantity of separate squares to form a cube means that the rigid cube can be totally decomposed: even if this cannot in itself be modified, the separate squares can represent the cube as if it were broken down into the parts of which it has been made up, giving rise to calculations and checks.

The number of beads making up each cube is the same as the number of beads forming the square, repeated as many times as there are units of the basic bead bar from which that square has originated (Figure 157).

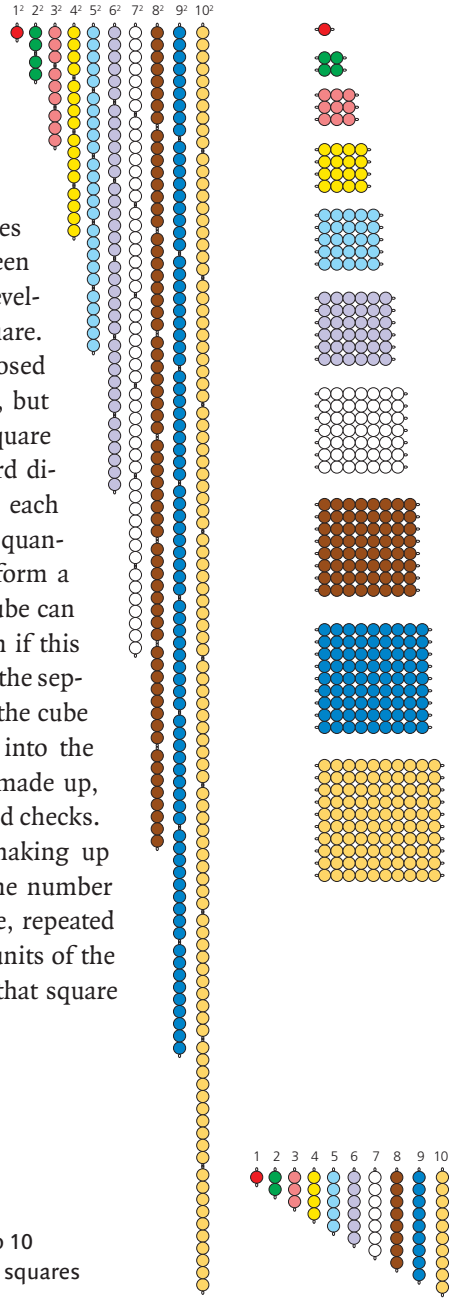

Figure 156
The squares of the numbers 1 to 10 represented by chains and rigid squares

the number is reached, a space[63] is left and bead bars continue to accumulate in columns. In this simple exercise, the children have found such interest that they prepared a tray with a velvet lining so that the bead bars do not slip. The layout of the material is shown in Figure 163.

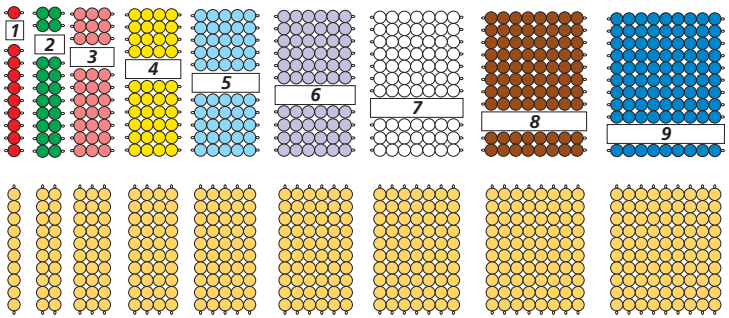

Figure 163
Multiplication by 10 with the coloured bead bars

The illustration reproduces as a whole all the numerical combinations constructed in the earlier exercises with the multiplication tables. The same products could be found by counting the units as they are being accumulated, whenever a new bar is added to the appropriate column. Nevertheless, placing all on the table provides a view that lends itself to making observations.

This is because when one takes each natural series number in turn, its square can be seen to be larger than the one before, while at the same time the height of the rectangle below is reduced. Below the unit bead is a column of nine beads. If we calculate the value of the squares, we find the numbers 1, 4, 9, 16, 25, 36, 49, 64, and 81: these are the numbers indicating the limits in the multiplication chart (Figure 79). The product of the whole group of beads in each column can easily be calculated because, since there are always ten rows of bars, the series of totals is 10, 20, ..., 70, 80 and 90. And, for this reason, bars of 10 corresponding to each of the products could be placed below each column. Beyond this

63 The cards with the numbers are inserted in the space after the square, although not mentioned in the text. They are, however, in the original Spanish illustration, which is why the illustration has been reproduced here.

would be the square of 10 which is different from all the groups because it is whole, while up to 9 groups have been able to be divided into two parts. Here is, then, because the square of 10 is out of character with the other groups considered, a unit of a higher order.

If one wants to calculate the value of the rectangle that is below each square, this is done either by multiplying or by subtracting, or also by combining the two operations. For example, if we look at the layout for the column of 4, the rectangle equals $4 \times 6 = 24$ beads, but it is also equal to $40 - 4^2 = 40 - 16 = 24$. By adding the square of each number to its rectangle, the result for each column will be:

$$
\begin{aligned}
10 &= 1 + 9 \\
20 &= 4 + 16 \\
30 &= 9 + 21 \\
40 &= 16 + 24 \\
50 &= 25 + 25 \\
60 &= 36 + 24 \\
70 &= 49 + 21 \\
80 &= 64 + 16 \\
90 &= 81 + 9
\end{aligned}
$$

After calculating the values and compiling the table, it can be observed that the rectangles represent amounts symmetrical around the square of 5. The square, therefore, is always the centre of symmetry. But the rectangles formed have a different construction, even though their values are the same. One of the rectangles with the value of 24, for example, is made up of six bars of 4, while the other is formed of four bars of 6. Where the value is 21, one rectangle consists of the bar of 3 repeated seven times, while the other consists of the bar of 7 repeated three times. We then come to a 16 made up of eight bars of 2, while another rectangle of 16 consists of two bars of 8. Lastly, there is a 9 consisting of nine individual beads and another 9 where the beads are laid out as a bar. The position of each rectangle in a pair, therefore, is rotated with respect to the position of the other. While the base of the rectangle is small and the height is greater in one case, the corresponding rectangle will have a wide base and its height will be low. If, however, each pair of rectangles is isolated by removing them from their context, a physical check can be made to show that they are equal. It is easy to calculate the numerical differences between successive rectangles: